ON THE HILBERT SCHEME COMPACTIFICATION OF
THE SPACE OF TWISTED CUBICS

by

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1. Introduction

One of the enumerative problems treated by Schubert in his book "Kalkul der abzählenden Geometrie" [S] is that of determining the number of twisted cubic curves which satisfy various given conditions. The complete solution to this problem should contain a description of the intersection ring of some compactification of the space of twisted cubics. In this paper we make a step in this direction by undertaking a study of the compactification given by the Hilbert scheme (see also [P]).

A twisted cubic curve is a rational, smooth curve of degree 3 in $\mathbb{P}^3$. The space H_o of such curves has the structure of a smooth, 12-dimensional, noncompact variety - in fact, H_o can be identified with the homogeneous space $SL(4)/SL(2)$. Let $\mathrm{Hilb}^{P(m)}(\mathbb{P}^3)$ denote the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^3$ with Hilbert polynomial $P(m)$. Then $H_o \subset \mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$, and we denote by H the closure of H_o. Our main result is the following theorem.

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<u>THEOREM</u>: $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ consists of two irreducible components, H and H', of dimension 12 and 15 respectively. Both H and H' are smooth and rational, they intersect transversally, and their intersection is non-singular, rational, of dimension 11.

The component H' which does not contain the twisted cubics contains the points corresponding to plane cubic curves union a point in $\mathbb{P}^3$. The intersection $H \cap H'$ consists of plane, singular cubic curves, with a "spatial" embedded point at a singular point, "emerging from" the plane. The most degenerated such curve (in the sense that all curves corresponding to points in $H \cap H'$ specialize to one of that kind) consists of a line tripled in the plane, with a spatial embedded point. A main ingredient in the proof of the theorem is the explicit construction of the deformation space of such a curve. We use a comparison theorem which enables us to identify the deformation theory of a projective variety with that of its associated homogeneous ideal, provided that suitable linear systems on the variety are complete (§3). The degenerate curve has a G_m action and its universal deformation is easy to compute (§5).

2. Preliminary description of $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$

Let $C \subset \mathbb{P}^3_k = \mathbb{P}^3$ (k is an algebraically closed field of characteristic $\neq 2,3$) be a twisted cubic curve, i.e., C is smooth, rational, of degree 3. All such curves are projectively equivalent, hence we may fix one, say $C_0 = \phi(\mathbb{P}^1)$, where $\phi: \mathbb{P}^1 \to \mathbb{P}^3$ is given by $\phi(u,v) = (u^3, u^2v, uv^2, v^3)$, and identify the space H_o of twisted cubics with automorphisms of $\mathbb{P}^3$ modulo automorphisms of $\mathbb{P}^1$. So $H_o = SL(4)/SL(2)$ is a homogeneous space, hence smooth and irreducible, of dimension 12.

Since a twisted cubic curve has Hilbert polynomial $P(m) = 3m+1$, we have $H_o \subset \mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$; let $H = \overline{H}_o$ denote its closure. Set $H_o' = \{C' : C' =$ a plane, smooth cubic curve in $\mathbb{P}^3$ union a point in $\mathbb{P}^3$ not on the curve $\}$; then $H_o' \subset \mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$, and we donote by $H' = \overline{H}_o'$ its closure. Since H_o' is irreducible, so is H', and H' has dimension 15.

<u>LEMMA 1</u>: $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3) = H \cup H'$.

<u>*Proof*</u>: Suppose $C \subset \mathbb{P}^3$ is a closed subscheme with Hilbert polynomial $P(m) = \chi(O_C(m)) = 3m+1$. We must show that C is a specialization of a curve in H_o or H_o'. Let $\overline{C} \subset C$ be the maximal closed subscheme of C which is Cohen-Macaulay and of pure dimension 1. There are three cases to consider: i) $\overline{C} = C$. Then C is projectively Cohen-Macaulay and there is a projective resolution of the maximal homogeneous ideal $I \subset P = k[x,y,z,w]$ defining C,

$$0 \to P(-3)^2 \to P(-2)^3 \to I \to 0 \quad [E, Ex.1, p. 430].$$

By [loc. cit, Thm. 2] C can be deformed to a twisted cubic. ii) $C = \overline{C} \cup Y$, where $Y \cap \overline{C} = \emptyset$ and $\lg O_Y = r \geq 1$. Since $\chi(O_C(m)) = \chi(O_{\overline{C}}(m)) + r = 3m+1$ and $\chi(O_{\overline{C}}(m)) \geq 3m$, we have $r = 1$ and $\chi(O_{\overline{C}}(m)) = 3m$. Hence $\overline{C}$ is a plane cubic curve, and Y is a reduced point, so $C \in H'$. iii) C has embedded points. Set $K = \mathrm{Ker}(O_C \to O_{\overline{C}})$. Reasoning as in the previous case, we conclude $\lg K = 1$, and $\overline{C}$ is plane, so that $C \in H'$.

<u>LEMMA 2:</u> If $C \in H \cap H'$, then C is a plane, singular cubic curve with a spatial embedded point, "emerging from" the plane, at a singular point. More precisely, C is projectively equivalent to the curve defined by an ideal $I \subset k[x,y,z,w]$ of the form $I=(xz,yz,z^2,q(x,y,w))$, where $q(x,y,w)$ is a cubic form which is singular at $(0,0,1)$.

<u>Proof:</u> With the notation of the proof of Lemma 1, $\bar{C}$ is plane and C is connected, so we're in case iii) of that proof. Moreover, it follows from a lemma of Hironaka [H, p. 360] that the embedded point must occur at a singular point of $\bar{C}$. It remains to describe the structure of C at the embedded point. First we observe that if C is contained in some surface $S \subset \mathbb{P}^3$, then S has to be singular at the embedded point p of C. In fact, we may assume that $C \subset S$ are the closed fibres of families $C_R \subset S_R \subset \mathbb{P}^3_R$, over a discrete valuation ring R with fraction field K, s.t. $C_K \subset \mathbb{P}^3_K$ is a twisted cubic (if deg $S=1$, replace S by S union a plane not containing p). If p were a smooth point on S, then it would be smooth on S_R, since S is a Cartier divisor on S_R. Then C_R would be a local complete intersection at p, hence so would C, and so p could not be an embedded point on C. Assume the embedded point is $(0,0,0,1)$, and that the ideal of C in the affine coordinate ring $k[x,y,z]$ is equal to $I_a=(z,q) \cap Q$, where $q(x,y)$ is singular at $(0,0)$, and Q is an (x,y,z)-primary ideal. Consider the exact sequence

$$0 \to K=(z,q)/I_a \to k[x,y,z]/I_a \to k[x,y]/(q) \to 0.$$

We know $\lg K=1$, so that either (a) $z \in Q$, or (b) there is a $q' \in Q$ with $q'=q \pmod z$. In case (a), $z \in I_a$, hence C is plane and cannot by the observation above, be the specialization of a twisted cubic. (In this case, $I=(z,xq,yq)$, and the ideal of C is obtained by homogenizing q with respect to w.) In case (b), $z \notin I_a$, but necessarily $(xz,yz,z^2,q') \subset I_a$, and these ideals are equal. Now $q'=q+\alpha z$, $\alpha \in k$, and if $\alpha \neq 0$, then the surface defined by q' would be smooth at $(0,0,0,1)$. By the observation above, we must therefore have $\alpha=0$ and hence

$q'=q$.

Note that it follows from Lemma 2, by counting parameters, that the dimension of $H \cap H'$ is equal to 11.

3. Local Description of the Hilbert Scheme.

To a subscheme X of $\mathbb{P}^n$ corresponds the homogeneous ideal I in the polynomial ring $P = k[x_0,\dots,x_n]$ such that $X=\mathrm{Proj}(P/I)$ and I is maximal with respect to this. We thus have a map

$$u : M \longrightarrow M'$$

from the universal deformation space M, which parametrizes all homogeneous ideals with Hilbert function equal to that of I, to the Hilbert scheme M' which parametrizes subschemes of $\mathbb{P}^n$ with Hilbert polynomial equal to that of X. We shall show here that M and M' are isomorphic near the base points I and X, provided that the linear systems cut out on X by hypersurfaces of suitable degrees are complete.

Comparison Theorem: If the ideal of polynomials defining $X \subset \mathbb{P}^n$ is generated by homogeneous polynomials $f_1,\dots,f_r$, of degrees $d_1,d_2,\dots,d_r$, for which

$$(k[x_0,\dots,x_n]/I)_d \xrightarrow{\sim} H^0(X,\mathcal{O}_X(d))$$

$$d = d_1,d_2,\dots,d_r ,$$

then the map $u : M \to M'$ is an analytic isomorphism at the basepoints I, X.

We remark that in general, when the completeness condition is not satisfied, one must replace I by a high truncation, as Curtin [C] does for Mumford's obstructed curve.

<u>Proof of the Comparison Theorem:</u> We compare the Zariski tangent and normal spaces of M and M'. Let $R = k[[t_1,\dots,t_m]]/J$, $J \subseteq (t)^2$, be the completion of the local ring of M at its base point. We have

$$t^1(M) = ((\underline{t})/(\underline{t})^2)^*$$

$$t^2(M) = (J/\underline{t}J)^* \quad ,$$

the Zariski tangent and normal (i.e. "obstruction") spaces of M (In general $t^i(M) = T^i(k/R,k)$ $i \geq 1$ are the "homotopy" of R [A].) Now u induces $u^i : t^i(M) \longrightarrow t^i(M')$, all i; as in [S,p.153] we find easily that $u : M \longrightarrow M'$ is an analytic isomorphism provided that u is a "two equivalence" in the sense that u^1 is an <u>isomorphism</u> and u^2 is a <u>monomorphism</u>.

If we now take $T^i(I) = T^i(A/P,A)$, $T^i(X) = T^i(X/\mathbb{P}^n,\mathcal{O}_X)$, the appropriate cotangent cohomology, we get a commutative diagram

$$\begin{array}{ccc} t^\cdot(M) & \longrightarrow & t^\cdot(M') \\ \downarrow & & \downarrow \\ T^\cdot(I) & \longrightarrow & T^\cdot(X) \end{array}$$

where vertical "Kodaira-Spencer" maps form are two equivalences, by versality of M and M'. We must show that $T^\cdot(I) \longrightarrow T^\cdot(X)$ is a two equivalence.

To compute $T^i(I)$ for $i = 1,2$ we take a free resolution

$$\cdots \longrightarrow H \xrightarrow{\nu} G \xrightarrow{\mu} F \xrightarrow{\lambda} P$$

of the module P/I over the polynomial ring $P = k[x_0,\dots,x_n]$.

Here $F = \Sigma\, P(-d_i)$ and $\lambda = (f_1,\ldots,f_r)$. We map $\Lambda^2 F \longrightarrow G$ by sending $u \wedge v$ to w in G with $\mu(w) = \lambda(u)v - \lambda(v)u \in \ker \lambda$. The <u>cotangent complex</u>, in low degrees, is then

$L : L_3 \longrightarrow L_2 \longrightarrow L_1 = \Lambda^2 F \otimes A \oplus H \otimes A \longrightarrow G \otimes A \longrightarrow F \otimes A$ with $A = P/I$ (see [L.S.]), and $T^i(I)$ is the cohomology of $L^\bullet = \mathrm{Hom}\,(L_\bullet, A)$.

Now the complex $\mathcal{L}_\bullet = \tilde{L}_\bullet$ restricts, over each affine open subset U of X to the relative cotangent complex of U in $\mathbb{P}^n$, so that $\mathcal{L}_\bullet$ is the cotangent complex of X in $\mathbb{P}^n$. Following Illusie [I] we then have $T^\bullet(X) = \mathrm{Ext}_{\mathcal{O}_X}(\mathcal{L}_\bullet, \mathcal{O}_X)$. If we consider instead the cohomology $S^\bullet(X)$ of the complex of vector spaces $\mathrm{Hom}(\mathcal{L}_\bullet, \mathcal{O}_X)$, the edge homomorphism $S^\bullet(X) \longrightarrow T^\bullet(X)$ is a two equivalence and we need only show that $T^\bullet(I) \longrightarrow S^\bullet(X)$ is a two equivalence. The map in question comes from taking cohomology of the horizontal rows of the diagram

$$\begin{array}{ccccc}
L^3_0 & \longleftarrow & L^2_0 & \longleftarrow & L^1_0 \\
\updownarrow{\scriptstyle \alpha_3} & & \updownarrow{\scriptstyle \alpha_2} & & \updownarrow{\scriptstyle \alpha_1} \\
H^0(\tilde{L}^3) & \longleftarrow & H^0(\tilde{L}^2) & \longleftarrow & H^0(\tilde{L}^1)
\end{array}$$

By hypothesis α_1 is an isomorphism, so that $\alpha_\bullet$ induces a two-equivalence and the proof is thus complete.

We remark that the cohomology sheaves $\mathcal{T}^i$ of $\underline{\mathrm{Hom}}\,(\mathcal{L}_\bullet, \mathcal{O}_X)$ consist of the normal sheaf $\mathcal{N} = \mathcal{T}^1$ to X in $\mathbb{P}^n$, which determines local deformations of X, and the sheaf $\mathcal{T}^2$ which contains obstructions to local deformations of X. $\mathcal{T}^2$ is supported on the non complete intersection locus of X. We have $H^0(X,\mathcal{T}^1) \cong T^1(X)$ and $0 \longrightarrow H^1(X,\mathcal{T}^1) \longrightarrow T^2(X) \longrightarrow H^0(X,\mathcal{T}^2)$ which decomposes $T^2(X)$ into local and global obstructions.

The hypotheses of the comparison theorem are certainly met for the smooth space curves in H_0 or H_0'. For a curve C in $H \cap H'$ let $I = (xz, yz, z^2, q)$, $J = (z,q)$ be the ideal of the Cohen-Macaulay curve $\overline{C}$ and $K = J/I$, which is isomorphic to $P/(x,y,z)$ twisted once as a P module. The local cohomology sequence associated to the exact sequence $0 \longrightarrow K \longrightarrow P/I \longrightarrow P/J \longrightarrow 0$ now shows that $(P/I)_d \longrightarrow H^0(\mathcal{O}_C(d))$ is an isomorphism for all $d > 0$. By the comparison theorem above we find that the completion of the Hilbert scheme at the point C is given by the universal deformation of the ideal I associated to C.

Alternatively, we may show directly that deformations of I and C agree by computing that the tangent space $T^1(I) = H^0(C,N_C) = H^0(\mathcal{T}^1)$ has dimension 16 and consists entirely of non positively weighted (thus globalizable) deformations of the singular point. Moreover, $H^1(\mathcal{T}^1) = 0$ and $T^2(I) = H^0(\mathcal{T}^2) \cong T^2(X)$ (has dimension two). The deformations of C thus coincide with the non-positive deformations of its singular point, which coincide with homogeneous deformations of the affine cone over C.

4. The tangent spaces to $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$.

Let $C \in \mathrm{Hilb}^{3m+1}(\mathbb{P}^3) = H \cup H'$, let $I \subset P = k[x,y,z,w]$ denote the maximal homogeneous ideal defining C, and set $A = P/I$, so that $\mathcal{O}_C = \tilde{A}$. Set $\mathcal{I} = \tilde{I}$, so that $\mathcal{N} = \underline{\mathrm{Hom}}(\mathcal{I}, \mathcal{O}_C) = (\mathcal{I}/\mathcal{I}^2)^{\vee}$ is the normal sheaf of C in $\mathbb{P}^3$. With this notation the tangent space to $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ at C is given by $T_{H \cup H', C} = H^0(C, \mathcal{N})$, and we now want to compute this space, which, as we have seen in §3, is isomorphic to $T^1(I) = \mathrm{Hom}_P(I,A)_0$ (the degree 0 piece of the graded module $\mathrm{Hom}_P(I,A)$.) By [E, loc. cit] we know that $H - H \cap H'$ is smooth, so that $\dim T_{H \cup H', C} = 12$ if $C \in H - H \cap H'$. (This can also be computed directly from a presentation of I, as will be done below in the other cases.)

<u>LEMMA 3</u>: If $C \in H \cap H'$, then $\dim T_{H \cup H', C} = 16$.

<u>Proof</u>: We may assume C is defined by a homogeneous ideal I as in Lemma 2. It suffices to show $\dim \mathrm{Hom}_P(I,A)_0 = 16$. Set $J = (z,q)$; then J defines a plane curve $\bar{C} \subset C$. Set $\bar{A} = P/J$ and $K = J/I$; then we have an exact sequence $0 \to K \to A \to \bar{A} \to 0$. Consider the following presentation of J:

$$0 \to P(-4) \to P(-1) \oplus P(-3) \to J \to 0.$$

By applying $\mathrm{Hom}_P(-,A)$ we obtain a long exact sequence which yields $\mathrm{Hom}_P(J,A) = A(1) \oplus M(3)$ (where $M = (x,y,z)A$) and $\mathrm{Ext}^1_P(J,A) = \bar{A}(4)$.

The presentation of K,

$$0 \to P(-4) \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} P(-3)^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} P(-2)^3 \xrightarrow{(x,y,z)} P(-1) \xrightarrow{z} K \to 0$$

shows $\mathrm{Hom}_P(K,A) = K(1)$ and that $\mathrm{Ext}^1_P(K,A)_0 \subset A(2)^3_0 = A^3_2$ is generated by $\left\{ \begin{pmatrix} zw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ zw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zw \end{pmatrix}, \begin{pmatrix} -q_1 \\ -q_2 \\ 0 \end{pmatrix} \right\}$, where

$q = xq_1 + yq_2$ (and $q_1, q_2 \in (x,y)A$, since q is singular at $(0,0,1)$.

From $0 \to I \to J \to K \to 0$ we therefore obtain the following exact sequence:

$0 \to K(1) \to A(1) \oplus M(3) \to \mathrm{Hom}_P(I,A) \to \mathrm{Ext}^1_P(K,A) \xrightarrow{\beta} \bar{A}(4)$. A diagram chase shows that the map β is the restriction of the map $(q_1, q_2, 0): A(2)^3 \to \bar{A}(4)$, and hence is 0. Thus we obtain a short exact sequence

$$0 \to \bar{A}(1) \oplus M(3) \to \mathrm{Hom}_P(I,A) \to \mathrm{Ext}^1_P(K,A) \to 0,$$

which yields $\dim \mathrm{Hom}_P(I,A)_0 = \dim \bar{A}_1 + \dim M_3 + \dim \mathrm{Ext}^1_P(K,A)_0 = 3+9+4 = 16$.

<u>LEMMA 4</u>: If $C \in H'-H \cap H'$, then $\dim T_{H \cup H', C} = 15$.

<u>Proof</u>: Case (i): $C = \bar{C} \cup Y$, where Y is a reduced point not on $\bar{C}$. Then $H^0(C,N) = h^0\left(\bar{C}, \mathcal{O}_{\bar{C}}(3) \oplus \mathcal{O}_{\bar{C}}(1)\right) + 3 = 15$. Case (ii): C has an embedded point emerging from the plane of $\bar{C}$, at a nonsingular point of $\bar{C}$. We may assume $I = (xz, yz, z^2, q)$, where $q \in k[x,y,w]$ is a cubic form which goes through, but is nonsingular at, the point $(0,0,1)$. The computation of its dimension is similar to the one above, except that in this case $\mathrm{Ext}^1_P(K,A)_0$ is generated by $\begin{pmatrix} zw \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ zw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ zw \end{pmatrix}$, which completes the count. Case (iii): C has an embedded point and is contained in a plane, i.e., we are in case (a) of the proof of Lemma 2. Then we may assume $I = (z, xq, yq)$, where $q \in k[x,y,w]$ is a cubic form vanishing at $(0,0,1)$. Set $P' = k[x,y,w]$ and $I' = (xq, yq) \subset P'$, and let N' denote the normal sheaf of C in the plane $z=0$. Reasoning as in the Comparison Theorem one shows $H^0(C,N') = \mathrm{Hom}_{P'}(I',A)_0$. Since $h^0(C,N) = h^0(C,N') + h^0(C,\mathcal{O}_C(1))$ and $h^0(C,\mathcal{O}_C(1)) = 4$, it suffices to show $\dim \mathrm{Hom}_{P'}(I',A)_0 = 11$.

Set $J' = (q) \subset P'$, $\bar{A} = P'/J'$, and $K = J'/I'$. The P'-module K has a presentation

$$0 \to P'(-5) \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} P'(-4)^2 \xrightarrow{(x,y)} P'(-3) \xrightarrow{q} K \to 0.$$

From this, we obtain $\mathrm{Hom}_{P'}(K,A) = K(3)$, and that $\mathrm{Ext}^1_{P'}(K,A)_0$ is generated by $\begin{pmatrix} qw \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ qw \end{pmatrix}$. Thus there is an exact sequence

$$0 \to \bar{A}(3) \to \mathrm{Hom}_{P'}(I',A) \to \mathrm{Ext}^1_P(K,A) \to 0,$$

and we conclude: $\dim \mathrm{Hom}_{P'}(I',A)_0 = \dim \bar{A}_3 + 2 = 11$.

Thus we have shown that $H \cup H'$ is smooth outside $H \cap H'$, and that $\dim T_{H\cup H',C} = 16$ if $C \in H \cap H'$.

<u>LEMMA 5</u>: For all $C \in \mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$, $H^1(C,N) = 0$.

<u>Proof</u>: We shall consider separately four cases and show that in each case we have $\chi(N) = h^0(C,N)$, by the above computations. i) $C \in H - H \cap H'$. The exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{I} \to 0$$

gives

$$0 \to N \to \mathcal{O}_C(2)^3 \to \mathcal{O}_C(3)^2 \to \omega_C(4) \to 0,$$

where ω_C denotes the dualizing sheaf on C, from which we get $\chi(N) = 12 = h^0(C,N)$. Hence $h^1(C,N) = 0$. ii) $C \in H \cap H'$. The exact sequence

$$0 \to \mathcal{O}_{\bar{C}}(1) \oplus \tilde{M}(3) \to N \to \underline{\mathrm{Ext}}^1_{\mathbb{P}^3}(K,\mathcal{O}_C) \to 0$$

gives $\chi(N) = 7 + \chi(\tilde{M}(3)) = 7 + \chi(\mathcal{O}_C(3)) - 1 = 16 = h^0(C,N)$. iii) $C \in H' - H \cap H'$, $C = \bar{C} \cup Y$, $Y \cap \bar{C} = \emptyset$. Then $\chi(N) = \chi(N_{\bar{C}/\mathbb{P}^3}) + 3 = 15 = h^0(C,N)$.
iv) $C \in H' - H \cap H'$, $C \subset P^2 \subset \mathbb{P}^3$, C has an embedded point. Then $\chi(N) = \chi(N_{C/\mathbb{P}^2}) + \chi(\mathcal{O}_C(1))$. The exact sequence (Lemma 4, (iii))

$$0 \to \mathcal{O}_{\bar{C}}(3) \to N_{C/\mathbb{P}^2} \to \underline{\mathrm{Ext}}^1_{\mathbb{P}^2}(\tilde{K},\mathcal{O}_C) \to 0$$

gives $\chi(N_{C/\mathbb{P}^2}) = 11$, hence $\chi(N) = 15 = h^0(C,N)$.

5. The universal deformation of $k[x,y,z,w]/(xz,yz,z^2,x^3)$.

From the description given in Lemma 2, we know that every $C \in H \cap H'$ specializes to a curve of the form: a line tripled in a plane, with a spatial embedded point. Such a curve is completely determined by its associated (point-line-plane) flag, and all such curves are projectively equivalent. (They form a closed orbit - isomorphic to the flag variety, hence of dimension 6 - under the action of $SL(4)$ on $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$.) Thus, in order to study deformations of some $C \in H \cap H'$, it suffices to study deformations of a curve of the above degenerated form, e.g. whose maximal homogeneous ideal is $I = (xz,yz,z^2,x^3)$.

LEMMA 6: Suppose $I = (xz,yz,z^2,x^3)$. Then I has a universal deformation space of the form $M = \mathbb{A}^{12} \cup \mathbb{A}^{15}$, where $\mathbb{A}^{12} \cap \mathbb{A}^{15} = A^{11}$ and the intersection is transversal.

Proof: Consider the following presentation of $A = P/I$ over $P = k[x,y,z,w]$:

$$0 \to P(-4) \xrightarrow{\nu} P(-4) \oplus P(-3)^3 \xrightarrow{\mu} P(-2)^3 \oplus P(-3) \xrightarrow{\lambda} P \to A \to 0,$$

where the maps are given by

$$\lambda = (xz,yz,z^2,x^3), \qquad \mu = \begin{pmatrix} x^2 & y & z & 0 \\ 0 & -x & 0 & z \\ 0 & 0 & -x & -y \\ -z & 0 & 0 & 0 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 0 \\ z \\ -y \\ x \end{pmatrix}$$

We have already seen (Lemma 3) that $\dim \mathrm{Hom}(I,A)_0 = 16$, and one checks that the following 10 elements (of $A_2^3 \oplus A_3$),

$$\frac{\partial}{\partial u_1} = \begin{pmatrix} 0 \\ x^2 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial u_2} = \begin{pmatrix} 0 \\ 0 \\ zw \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial u_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ xw^2 \end{pmatrix}, \quad \frac{\partial}{\partial u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ yw^2 \end{pmatrix}, \quad \frac{\partial}{\partial u_5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ zw^2 \end{pmatrix},$$

$$\frac{\partial}{\partial u_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x^2w \end{pmatrix}, \quad \frac{\partial}{\partial u_7} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ xyw \end{pmatrix}, \quad \frac{\partial}{\partial u_8} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^2w \end{pmatrix}, \quad \frac{\partial}{\partial u_9} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ xy^2 \end{pmatrix}, \quad \frac{\partial}{\partial u_{10}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^3 \end{pmatrix},$$

together with the 6 "trivial deformations" (corresponding to moving the flag determined by C),

$$\frac{\partial}{\partial t_1} = w\frac{\partial}{\partial x} = \begin{pmatrix} zw \\ 0 \\ 0 \\ 3x^2w \end{pmatrix}, \quad \frac{\partial}{\partial t_2} = y\frac{\partial}{\partial x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3x^2y \end{pmatrix}, \quad \frac{\partial}{\partial t_3} = w\frac{\partial}{\partial y} = \begin{pmatrix} 0 \\ zw \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial t_4} = w\frac{\partial}{\partial z} = \begin{pmatrix} xw \\ yw \\ 2zw \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial t_5} = x\frac{\partial}{\partial z} = \begin{pmatrix} x^2 \\ xy \\ 0 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial t_6} = y\frac{\partial}{\partial z} = \begin{pmatrix} xy \\ y^2 \\ 0 \\ 0 \end{pmatrix},$$

form a basis of $\mathrm{Hom}(I,A)_0$.

To obtain homogeneous deformations of I, we consider homogeneous perturbations λ', μ', ν' of the maps λ, μ, ν :

$$\lambda' = (xz+u_1(bx+cy),\ yz-u_1(ax+by),\ (z+u_2w)z-u_1^2(b^2-ac),$$
$$ax^2+2bxy+cy^2+(u_3x+u_4y+u_5(z+u_2w))w^2),$$

$$\mu' = \begin{pmatrix} ax+by+u_3w^2 & y & z-u_1b+u_2w & u_1a \\ bx+cy+u_4w^2 & -x & -u_1c & z+u_1b+u_2w \\ u_5w^2 & 0 & -x & -y \\ -z & -u_1 & 0 & 0 \end{pmatrix}, \quad \nu' = \begin{pmatrix} -u_1 \\ z+u_2w \\ -y \\ x \end{pmatrix}.$$

where we have set $a = x+u_6w$, $b = u_9y+u_7w$, $c = u_{10}y+u_8w$, and where the variables $u_i, i=1,\ldots,10$ give infinitesimal deformations tangent to the basis elements of $\mathrm{Hom}_P(I,A)_0$ denoted by $\frac{\partial}{\partial u_i}$.

One checks that $\lambda' \cdot \mu' \equiv 0 \mod (u_1u_2,u_1u_3,u_1u_4,u_1u_5)$. Moreover, $\mu' \cdot \nu' = (-u_1u_3w^2,-u_1u_4w^2,-u_1u_5w^2,-u_1u_2w) \equiv 0 \mod (u_1u_2,u_1u_3,u_1u_4,u_1u_5)$, and no additional higher order terms can cancel these entries. Therefore, the flat deformation that we can obtain over the union of the 6-space $u_2 = u_3 = u_4 = u_5 = 0$ with the 9-space $u_1 = 0$ cannot be extended to any larger parameter space. (Alternatively, the entries arising from $\lambda' \cdot \mu'$ may be shown to span T^2.) We have thus exhibited a versal deformation of I (and hence of C).

A universal deformation is now obtained from the above by adding the trivial deformations; this is done by performing everywhere the following substitutions: $x = x+t_1w+t_2y$, $y = y+t_3w$, $z = z+t_4w+t_5x+t_6y$. Hence we have shown:

$$M = \mathrm{Spec}(k[u_1,\ldots,u_{10},t_1,\ldots,t_6]/(u_1u_2,u_1u_3,u_1u_4,u_1u_5)).$$

<u>REMARK:</u> Recalling the exact sequence (proof of Lemma 3)

$$0 \to \bar{A}(1) \oplus M(3) \to \mathrm{Hom}(I,A) \to \mathrm{Ext}^1(K,A) \to 0$$

and remarking that $a \in \bar{A}_1$ goes to $\begin{pmatrix} xa \\ ya \\ za \\ 0 \end{pmatrix} \in \mathrm{Hom}(I,A)_0 \subset A_2{}^3 \oplus A_3$

and $b \in M_3$ goes to $\begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix}$, we observe that $\mathrm{Hom}(I,A)_0$ is generated, modulo the trivial deformations, by the elements coming from M_3, together with the elements $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$. The former corresponds to "twisting the curve into space" (hence making the embedded point disappear), whereas the latter corresponds to moving the embedded point out of the plane.

6. The Hilbert Scheme $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$.

We shall now prove the theorem stated in the introduction.

Theorem: The scheme $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ is the union of two nonsingular rational varieties H and H', of dimension 12 and 15 ; their transversal intersection is nonsingular rational of dimension 11.

Proof: By the previous Lemmas, we need only demonstrate the rationality of H, H', $H \cap H'$. Consider the point $C_0 \in H \cap H'$ whose ideal is $I = (xz, yz, z^2, x^3)$, and the universal family of deformations of I constructed in § 5. We get a flat family $\mathcal{X} \longrightarrow \mathbb{A}^{12} \cup \mathbb{A}^{15}$ of subschemes of $\mathbb{P}^3$ and hence a classifying map $\phi : \mathbb{A}^{12} \cup \mathbb{A}^{15} \longrightarrow \mathrm{Hilb}(\mathbb{P}^3)$. We have seen that ϕ is an analytic isomorphism at each point of its domain. The ideal I occurs only at the base point of $A^{12} \cup A^{15}$, and does not reappear as the parameters (u,t) approach infinity. ϕ has degree one over the Hilbert point C_0, and therefore has degree one over each point of its image, as any such point specializes to C_0. ϕ is therefore an open immersion, and the theorem follows.

Alternatively, one may compute directly that the ideals corresponding to parameter values (u,t) and (u',t') are not equal unless $u = u'$, $t = t'$, and proceed as above.

Also, as Robert Varley has kindly pointed out to us, the rationality of H is classical. Fix two planes p_1, p_2 in $\mathbb{P}^3$. They intersect a general twisted cubic C in two pairs of three points, and these six points in turn determine C. H is thus birationally equivalent to the product of $\mathrm{Symm}_3(\mathbb{P}^2)$ with itself. A modern proof of the rationality of a symmetric product was given by Mattuck [M]; this may also be seen from the versal deformation of a suitable thick point.

Corollary: The scheme H decomposes as a finite disjoint union of affine spaces, $H = \mathbb{A}^{12} \cup \bigcup \mathbb{A}^{n_i}$, where $0 \leq n_i \leq 11$ and all integers between 0 and 11 occur.

Proof: H is smooth, complete (in fact projective), and has a finite number of orbits under the action of $SL(4)$. By a result of D. Luna (see [D-P], 7.2) the set of fixed points of a maximal torus of $SL(4)$ is finite, and therefore we can apply the results of Bialynicki-Birula [B,§4]. (The $\mathbb{A}^{12}$ found in the proof of the theorem can be taken as the beginning of such a decomposition.)

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